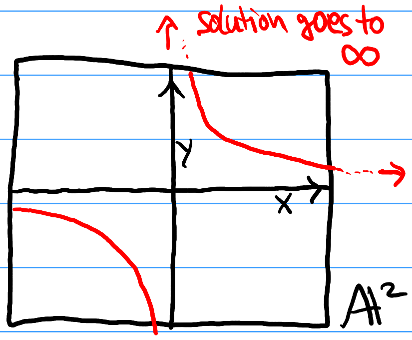
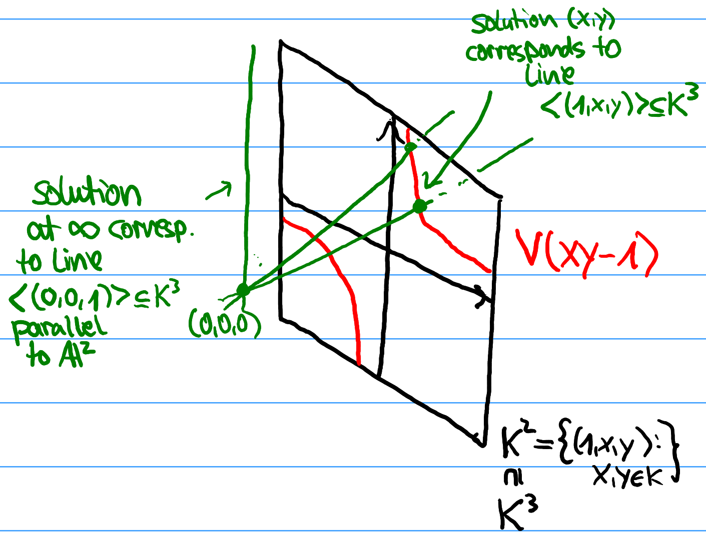
Projective varieties I : Topology

Big picture

- affine varieties = closed subvarieties of $\mathbb{A}^n$ (*)
- to get more general varieties : can glue affine varieties "by hand"
  Disadvantages:
  - lots of data $(U_i, U_{ij} \xrightarrow{f_{ij}} U_{ji})$
  - check compatibilities
  - check separatedness (how?)
- Nicer approach to create many examples:
  construct one nice ambient variety $\mathbb{P}^n$ (projective space)
  $\rightsquigarrow$ projective varieties = closed subvarieties of $\mathbb{P}^n$ $\leftarrow$ similar to (*)
- these will cover lots of interesting examples
  (actually: hard to construct variety which is not loc. closed in some $\mathbb{P}^n$)
- this chapter & next : develop language for $\mathbb{P}^n$ from first principles ($\rightsquigarrow$ useful for concrete computations)

Motivation





Def (Projective space)
For $n \in \mathbb{N}$ define the <u>projective n-space</u> $\mathbb{P}^n = \mathbb{P}^n_K$ as the set

$$\mathbb{P}^n = \{\ell \leq K^{n+1} : \ell \text{ is 1-dim'l linear subspace of } K^{n+1}\}$$

Coordinates on projective space

Want: concrete way to describe points of $\mathbb{P}^n$ using numbers (like coordinates in $\mathbb{A}^n$)

Notation (Homogeneous coordinates)

→ any 1-dim'l space $\ell \leq K^{n+1}$ can be specified by giving a generator

$$\ell = \langle \underbrace{(x_0, x_1, \ldots, x_n)}_{\underline{x} \in K^{n+1} \setminus \{0\}} \rangle$$

→ two vectors $\underline{x}, \underline{y}$ span the same space if they are lin. dependent ⟺ scalar multiples of one another

$$\Rightarrow \mathbb{P}^n = (K^{n+1} \setminus \{0\}) / \sim \text{ with equivalence relation}$$

$$\underline{x} = (x_0, \ldots, x_n) \sim \underline{y} = (y_0, \ldots, y_n) \iff \exists \lambda \in K^* : x_i = \lambda y_i \ \forall i$$

$K^* = K \setminus \{0\}$ mult. group

Write: $\mathbb{P}^n = (K^{n+1} \setminus \{0\}) / K^*$

equivalence classes:

other notations: $[x_0 : \ldots : x_n]$ or $[x_0, \ldots, x_n]$

$$(x_0 : x_1 : \ldots : x_n) = [(x_0, \ldots, x_n)]$$

numbers x_i = <u>homogeneous coordinates</u>

not all 0   only defined up to simultan. scaling

Exa $(5 : -2 : 3) = (15 : -6 : 9) = (1 : -\frac{2}{5}, \frac{3}{5}) \in \mathbb{P}^2$

Affine space as a subset of projective space

Rmk (Affine coordinates)

Consider the map

$$\varphi : \mathbb{A}^n \longrightarrow \mathbb{P}^n, \quad (x_1,\dots,x_n) \longmapsto (1 : x_1 : \dots : x_n)$$

set $x_0 = 1$

$\rightsquigarrow$ φ is injective : $\varphi(\underline{x}) = \varphi(\underline{y}) \Leftrightarrow \exists \lambda \in K^* : \lambda \cdot (1, x_1, \dots, x_n) = (1, y_1, \dots, y_n)$

$\Leftrightarrow \underline{x} = \underline{y}$

zeroth coord. forces $\lambda = 1$

$\rightsquigarrow$ Image of φ:

$$U_0 = \operatorname{im}(\varphi) = \{ (x_0 : x_1 : \dots : x_n) : x_0 \neq 0 \}$$

$\wr$

$(1 : \frac{x_1}{x_0} : \dots : \frac{x_n}{x_0})$

$\rightsquigarrow$ Inverse of φ:

$$\varphi^{-1} : U_0 \longrightarrow \mathbb{A}^n, \quad (x_0 : x_1 : \dots : x_n) \longmapsto \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0} \right).$$

affine coord.

$\Rightarrow$ Think of $\mathbb{A}^n = U_0 \subseteq \mathbb{P}^n$ as subset (affine part of $\mathbb{P}^n$)

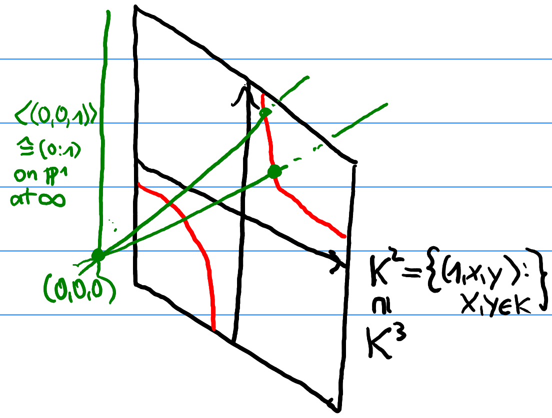
$\rightsquigarrow$ Complement

$$\begin{array}{ccc} \mathbb{P}^n \setminus \mathbb{A}^n = \{ (0 : x_1 : \dots : x_n) : (x_1,\dots,x_n) \in K^n \setminus \{0\} \} \\ \downarrow \wr \\ \mathbb{P}^{n-1} = \{ (x_1 : \dots : x_n) : (x_1,\dots,x_n) \in K^n \setminus \{0\} \} \end{array}$$

$$\Rightarrow \mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$

affine part

points at infinity



Rmk (Shortcut to projective space as variety)

- have $U_0 = \{(x_0 : x_1 : \dots : x_n) : x_0 \neq 0\} \cong \mathbb{A}^n$
- similar: $U_i = \{(x_0 : x_1 : \dots : x_n) : x_i \neq 0\} \cong \mathbb{A}^n$ for $i = 0, \dots, n$
- Then

  $\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$ gives a cover $((\underline{x}) \in \mathbb{P}^n \Rightarrow \exists i : x_i \neq 0)$
- Check: transition functions

$$\mathbb{A}^n \cong U_i \supseteq U_i \cap U_j \xrightarrow{\text{id}} U_i \cap U_j \subseteq U_j \cong \mathbb{A}^n$$

are morphisms & satisfy compatibility $\Rightarrow$ get prevariety $\mathbb{P}^n$ by gluing

- instead: develop new language (homog. coord., ideals, etc) & define ringed space structure on $\mathbb{P}^n$ "by hand".

Rmk (Complex projective space is compact).

For $K = \mathbb{C}$:

surjective map

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n_{\mathbb{C}}, \quad (x_0, \dots, x_n) \mapsto (x_0 : \dots : x_n)$$

give $\mathbb{P}^n_{\mathbb{C}}$ the quotient topology

Lem $\mathbb{P}^n_{\mathbb{C}}$ is compact

Pf Let $S = \{(x_0, \dots, x_n) \in \mathbb{C}^{n+1} : |x_0|^2 + \dots + |x_n|^2 = 1\} \subseteq \mathbb{C}^{n-1} \setminus \{0\}$

unit sphere

$\Rightarrow S$ compact and $\pi|_S : S \to \mathbb{P}^n_{\mathbb{C}}$ still surjective

can rescale any $(x_0, \dots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ to have norm 1.

$\Rightarrow \mathbb{P}^n_{\mathbb{C}} = \pi(S)$ is compact as continuous image of a compact set.

Homogeneous polynomials & graded rings

Big picture

- affine varieties = sol'ns of polynomial equations in coordinates of $\mathbb{A}^n$ (e.g. $X = V(x_1^2 - x_2) \subseteq \mathbb{A}^2$)
- cannot immediately do same thing with homogen. coordinate

<u>Problem</u> What is $\{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : x_1^2 - x_2 = 0\}$?

$(1:1:1) \rightsquigarrow 1^2 - 1 = 0$ ✓

$\overset{\shortparallel}{(-1:-1:-1)} \rightsquigarrow (-1)^2 - (-1) = 2 \neq 0$ ← char $K \neq 2$

$\rightsquigarrow$ solutions of $x_1^2 - x_2 = 0$ not independent of scaling!

- Solution: only consider <u>homogeneous polynomial</u> equations
  ↳ all monomials have same (total) degree d

Then: $f \in K[x_0, \dots, x_n]$ homogeneous of degree d

$$\Rightarrow f(\lambda \cdot x_0, \lambda \cdot x_1, \dots, \lambda \cdot x_n) = \lambda^d \cdot f(x_0, \dots, x_n)$$

In particular:

$$f(\lambda \cdot x_0, \lambda \cdot x_1, \dots, \lambda \cdot x_n) = 0 \Leftrightarrow f(x_0, \dots, x_n) = 0$$

↑ Zero locus of f well-defined in $\mathbb{P}^n$

<u>Exa</u> $f(x_0, x_1) = \underbrace{x_0^2 - x_0 x_1}_{= x_0 \cdot (x_0 - x_1)} \rightsquigarrow f(\lambda x_0, \lambda x_1) = (\lambda x_0)^2 - (\lambda x_0)(\lambda x_1) = \lambda^2 (x_0^2 - x_0 x_1)$

$$\Rightarrow \{(x_0 : x_1) \in \mathbb{P}^1 : f(x_0, x_1) = 0\} = \underbrace{\{(x_0 : x_1) : x_0 = 0\}}_{= \{(0:1)\}} \cup \underbrace{\{(x_0 : x_1) : x_0 = x_1\}}_{= \{(1:1)\}}$$

Correct formulation for more general rings than $K[x_0, \dots, x_n]$: graded rings!

Def (Graded rings and K-algebras)

(a) A <u>graded ring</u> is a ring R together with Abelian subgroups $R_d \subseteq (R,+)$ for all $d \in \mathbb{N} = \{0,1,2,\dots\}$ such that

- We have $R = \bigoplus_{d\in\mathbb{N}} R_d$, i.e. every $f \in R$ has a unique decomposition $f = \sum_{d\in\mathbb{N}} f_d$ such that $f_d \in R_d$ for $d \in \mathbb{N}$ and only finitely many f_d are non-zero.
- For all $d,e \in \mathbb{N}$ and $f \in R_d$, $g \in R_e$ we have $f \cdot g \in R_{d+e}$.

For $f \in R\setminus\{0\}$:

$$\deg(f) = \max\{d : f_d \neq 0\} \quad \underline{\text{degree}} \text{ of } f$$

Elements of $R_d \setminus \{0\}$: <u>homogeneous of degree d</u>

$f = \sum_{d\in\mathbb{N}} f_d$, $R = \bigoplus_{d\in\mathbb{N}} R_d$: <u>homogeneous decompositions</u>

(b) R as above also K-algebra:
R is a <u>graded K-algebra</u> if $\lambda f \in R_d$ for all $\lambda \in K$, $f \in R_d$

Exa Polynomial ring $R = K[x_0,\dots,x_n]$ is graded K-alg.

$R_0 = K = \mathrm{Span}_K\{1\}$, $R_1 = \mathrm{Span}_K\{x_0,\dots,x_n\}$

$R_2 = \mathrm{Span}_K\{x_0^2, x_0x_1, \dots, x_0x_n, x_1^2, x_1x_2, \dots\dots, x_n^2\}$

More generally:

$$R_d = \left\{ \sum_{\substack{i_0,\dots,i_n\in\mathbb{N}\\ i_0+\dots+i_n=d}} a_{i_0,\dots,i_n} \cdot x_0^{i_0}\cdots x_n^{i_n} : a_{i_0,\dots,i_n} \in K \text{ for all } i_0,\dots,i_n \right\}$$

$f = x_0 + 2x_0^2 + 3x_1x_2 + x_3^4 \rightsquigarrow$ $f_1 = x_0$, $f_2 = 2x_0^2 + 3x_1x_2$, $f_3 = 0$, $f_4 = x_3^4$

$\rightsquigarrow$ homog. decomposition

<u>Exercise</u> What part of def. goes wrong for $R = K[[x_0,\dots,x_n]]$?
Power series ring

<u>Exercise</u> Let $R \neq 0$ be a graded ring.

- Show that $1 \in R$ is homogeneous of degree 0.
- Conclude that grading on $K[x_0,\dots,x_n]$ uniquely determined by declaring $x_0,\dots,x_n$ homogeneous of degree 1.

more general grading: $\deg(x_i) = d_i \in \mathbb{N}$

Homogeneous ideals

We saw: $f \in K[x_0,\dots,x_n]$ homogeneous $\rightsquigarrow$ vanishing locus of f in $\mathbb{P}^n$ well-defined

Q What ideals of $K[x_0,\dots,x_n]$ have well-def. vanish. loci?

Def (Homogeneous ideals)
An ideal in a graded ring is called homogeneous if it can be generated by homogeneous elements.
↖ not necess. of same degree!

Exa In $R = K[x_0,x_1,x_2]$:

$I_1 = \langle x_0 - x_1,\ x_2^2 + 2x_0x_1 \rangle$ homogeneous

$I_2 = \langle x_0 - x_1^2 \rangle$ not homogeneous

Note:
Not every set of generators has to be homogeneous:

$$I_1 = \langle x_0 - x_1,\ \underbrace{x_2^2 + 2x_0x_1 + x_0 - x_1}_{\text{not homogeneous!}} \rangle$$

Lem (Properties of homogeneous ideals)
Let J, J_1, J_2 be ideals in a graded ring R.
(a) The ideal J is homogeneous if and only if for $f \in J$ with $f = \sum_{d\in\mathbb{N}} f_d$ we also have $f_d \in J$ $\forall d$.
(b) If J_1 and J_2 are homogeneous, then so are $J_1 + J_2$, $J_1 \cdot J_2$, $J_1 \cap J_2$ and $\sqrt{J_1}$.
(c) If J is homogeneous then the quotient R/J is a graded ring with hom. decomposition

$$R/J = \bigoplus_{d\in\mathbb{N}} R_d/(R_d \cap J).$$

Exa $J = \langle x^2 \rangle \subseteq K[x]$ homogeneous,
$f = (2+x)\cdot x^2 = 2x^2 + x^3 \in J \overset{(a)}{\Longrightarrow} f_2 = 2x^2$ and $f_3 = x^3$ also cont. in J.

Proof (a) Crucial observation: $\boxed{f,g\in R \Rightarrow (f\cdot g)_d = \sum_{\substack{d_1,d_2\in\mathbb{N}\\ d_1+d_2=d}} f_{d_1}\cdot g_{d_2}}$ (*)

"$\Rightarrow$" $J=\langle h^{(i)} : i\in I\rangle$ homog. ideal, $f\in J$
homogeneous

$$\Rightarrow f=\sum_{i\in I} g^{(i)}\cdot h^{(i)} \overset{(*)}{\Longrightarrow} f_d=\sum_{i\in I} g^{(i)}_{d-\deg(h^{(i)})}\cdot h^{(i)}\in J.$$

not necess. homog.
only fin. many nonzero terms $g^{(i)}$

"$\Leftarrow$" Assume $J\subseteq R$ ideal with ($f\in J \overset{(\star)}{\Rightarrow} f_d\in J\ \forall d$)

Claim

$$J=\langle f_d : f\in J,\ d\in\mathbb{N}\rangle$$

"$\supseteq$" by ($\star$), "$\subseteq$" for $f\in J \rightsquigarrow f=\sum f_d$ ← are in right-hand ideal.

(b) $J_1=\langle h^{(i)} : i\in I_1\rangle$, $J_2=\langle g^{(j)} : j\in I_2\rangle$, $h^{(i)}, g^{(j)}$ homogen.

$\Rightarrow J_1+J_2=\langle \{h^{(i)} : i\in I_1\}\cup\{g^{(j)} : j\in I_2\}\rangle$ homogen.

homog. generators

$J_1\cdot J_2=\langle\{h^{(i)}\cdot g^{(j)} : i\in I_1, j\in I_2\}\rangle$ homogen.

For $J_1\cap J_2$ use part (a): $f\in J_1\cap J_2 \Rightarrow f\in J_1$ and $f\in J_2$

$\overset{(a)}{\Longrightarrow} f_d\in J_1$ and $f_d\in J_2\ \forall d$

$\Rightarrow f_d\in J_1\cap J_2.$

$\Rightarrow$ $J_1\cap J_2$ hom.

Remains: $\sqrt{J_1}$ homogeneous. Let's prove criterion from (a):

Let $f=f_0+f_1+\dots+f_d\in\sqrt{J_1}$, then $\exists n\in\mathbb{N}$ such that

$$J_1\ni f^n=\underbrace{f_d^n}_{\substack{\text{hom. part of } f^n\\ \text{of degree } d\cdot n}}+(\text{terms of lower degree}) \overset{(a)}{\Longrightarrow} f_d^n\in J_1 \overset{\text{Def.}}{\Longrightarrow} f_d\in\sqrt{J_1}$$

But then $\tilde{f}=\underbrace{f}-\underbrace{f_d}=f_0+\dots+f_{d-1}\in\sqrt{J_1}$
both in $\sqrt{J_1}$

$\xrightarrow[\text{[Induct. on } d]]{\text{repeat argum.}}$ $f_i\in\sqrt{J_1}$ for all i $\overset{(a)}{\Longrightarrow}$ $\sqrt{J_1}$ homogeneous

(c) Want to show: $R/J = \bigoplus_{d\in\mathbb{N}} R_d / R_d \cap J$ graded ring.

homogeneous

- $R_d / R_d \cap J$ subgroup of R/J:

Indeed

$\varphi_d : R_d \longrightarrow R/J$ group morphism w/ kernel $R_d \cap J$

$f \longmapsto \bar{f}$

$\xRightarrow[\text{thm.}]{\text{Isom.}} R_d / R_d \cap J = \mathrm{im}(\varphi_d) \overset{\text{subgroup.}}{\subseteq} R/J$

- Existence of homog. decomp. in R/J:

$\bar{f} \in R/J \,,\; f = \sum_{d\in\mathbb{N}} f_d \Rightarrow \bar{f} = \sum_{d\in\mathbb{N}} \underbrace{\bar{f_d}}_{\in R_d / R_d \cap J}$

- Uniqueness of homo. decomp. in R/J:

$\sum_{d\in\mathbb{N}} \bar{f_d} = \sum_{d\in\mathbb{N}} \bar{g_d} \in R/J \Rightarrow \Delta = \sum_{d\in\mathbb{N}} f_d - g_d \in J$

$\overset{(a)}{\Longrightarrow} \Delta_d = f_d - g_d \in J \cap R_d$

$\Rightarrow \bar{f_d} = \bar{g_d} \in R_d / R_d \cap J.$ $\square$

Projective varieties

Idea Affine varieties = subsets of $\mathbb{A}^n$ cut out by polynomial equns.
Projective varieties = " " $\mathbb{P}^n$ " " " homogeneous " " .

Def (Projective varieties and their ideals)
(a) Let $S \subseteq K[x_0, \ldots, x_n]$ be a set of homogen. polynomials.
Then the (projective) zero locus (or vanishing locus) of S is defined as

$$V(S) := \{ x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in S \} \subseteq \mathbb{P}^n$$

well-defined

Subsets $V(S) \subseteq \mathbb{P}^n$: projective varieties

(b) For a homogeneous ideal $J \trianglelefteq K[x_0, \ldots, x_n]$ we set

$$V(J) := \{ x \in \mathbb{P}^n : f(x) = 0 \text{ for all homogeneous } f \in J \} \subseteq \mathbb{P}^n$$

note: S homogeneous $\rightsquigarrow$ $V(\langle S \rangle) = V(S)$.

(c) $X \subseteq \mathbb{P}^n$ any subset $\rightsquigarrow$ define its ideal as

$$I(X) := \langle f \in K[x_0, \ldots, x_n] \text{ homogeneous} : f(x) = 0 \ \forall x \in X \rangle \trianglelefteq K[x_0, \ldots, x_n]$$

take ideal generated by those f.

Notation: I_p, V_p for projective constructions above
I_a, V_a for affine constructions (in $\mathbb{A}^{n+1}$) from chapt. 1

Exa (a) $\emptyset = V_p(1)$, $\mathbb{P}^n = V_p(0)$ are proj. varieties
(b) $f_1, \ldots, f_r \in K[x_0, \ldots, x_n]$ homogeneous, linear polynomials
$\Rightarrow V_p(f_1, \ldots, f_r) \subseteq \mathbb{P}^n$ called a linear subspace of $\mathbb{P}^n$

Exercise $a \in \mathbb{P}^n \rightsquigarrow$ Show that $\{a\}$ is a proj. variety
and compute explicit generators of $I_p(\{a\})$.

Exa $f = x_1^2 - x_2^2 - x_0^2 \in \mathbb{C}[x_0, x_1, x_2]$

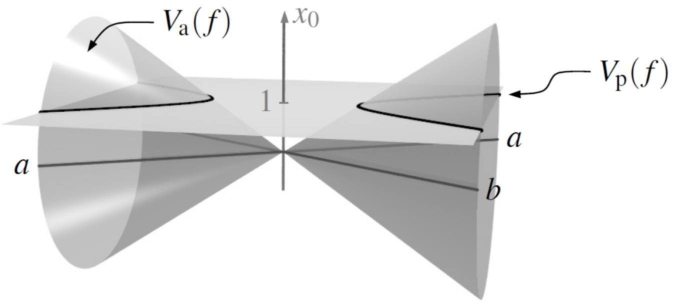
$\rightsquigarrow V_a(f) \subseteq \mathbb{A}^3$ (real points shown on bottom left)

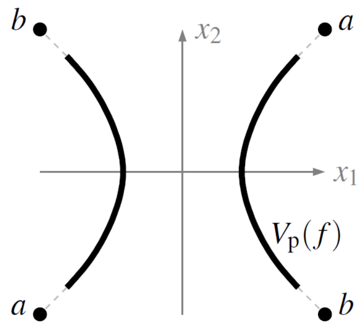
$V_p(f) \subseteq \mathbb{P}^2$ is set of 1-dim. lin. subspaces of $\mathbb{A}^3$ cont. in $V_a(f)$

Have seen: $\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{P}^1$ with $\mathbb{A}^2 \xrightarrow{i} \mathbb{P}^2$, $(x_1, x_2) \mapsto (1 : x_1 : x_2)$

$\rightsquigarrow i^{-1}(V_p(f)) = V_a^{\mathbb{A}^2}(x_1^2 - x_2^2 - 1)$ hyperbola on bottom right

$\cong V_a^{\mathbb{A}^3}(x_1^2 - x_2^2 - x_0^2, x_0 - 1)$



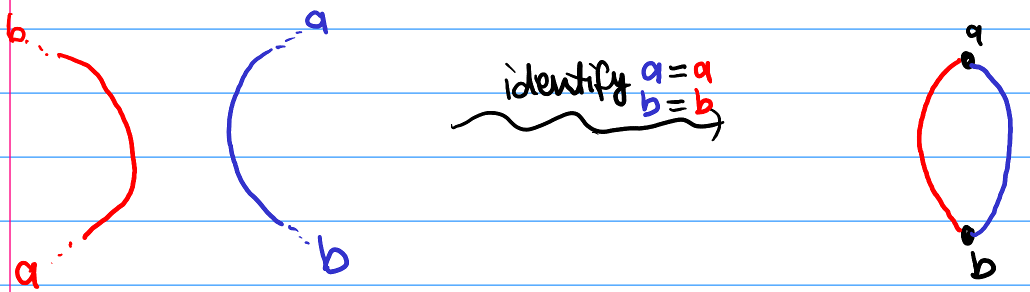


Missing points on $\mathbb{P}^1 = \{(x_0 : x_1 : x_2) : x_0 = 0\}$:

$x_1^2 - x_2^2 = 0 \Leftrightarrow x_1 = x_2 \vee x_1 = -x_2$

$a = (0:1:1)$ $\quad b = (0:1:-1)$ } see lines on left

In fact: real points of $V_p(f)$ = circle



Note Above example shows that $V_p(f)$ and $V_a(f)$ carry basically same information

$\rightsquigarrow$ make precise in next section

Cones & projective varieties

Big picture
- For S homogeneous: $V_a(S)$ and $V_p(S)$ encode similar info
- Below: make this precise ⇝ can import lots of results for affine variety $V_a(S)$ to describe $V_p(S)$.

Def (Cones)
Let $\pi: \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be projection $(x_0, \dots, x_n) \mapsto (x_0 : \dots : x_n)$.

(a) Affine variety $X \subseteq \mathbb{A}^{n+1}$ called a <u>cone</u> if
$\underbrace{0 \in X}_{\overset{\text{second}}{\underset{\text{condition}}{\Longleftrightarrow}} X \neq \emptyset}$ and $\lambda x \in X$ for all $\lambda \in K$, $x \in X$

(b) For $X \subseteq \mathbb{A}^{n+1}$ a cone:

$$\mathbb{P}(X) := \pi(X \setminus \{0\}) \qquad \underline{\text{projectivization}} \text{ of } X.$$
$$= \{(x_0 : \dots : x_n) \in \mathbb{P}^n : (x_0, \dots, x_n) \in X\} \subseteq \mathbb{P}^n$$

(c) For $X \subseteq \mathbb{P}^n$ projective variety:

$$C(X) := \{0\} \cup \pi^{-1}(X) \qquad \underline{\text{cone}} \text{ over } X$$
$$= \{0\} \cup \{(x_0, \dots, x_n) : (x_0 : \dots : x_n) \in X\} \subseteq \mathbb{A}^{n+1}$$

↳ show below that this is a cone in sense of (a).

Rmk (Cones and homogeneous ideals)
(a) $S \subseteq K[x_0, \dots, x_n]$ set of non-const. homog. polynomials
$\Rightarrow V_a(S)$ is a cone

Proof Every non-const. hom. polynomial vanishes at $0 \Rightarrow 0 \in V_a(S)$
For $\lambda \in K$, $x \in V_a(S)$ ⇝ $f(x) = 0 \ \forall f \in S$ ⇝ $0 = \lambda^{\deg(f)} \cdot f(x) = f(\lambda \cdot x)$ $\forall f \in S$
⇝ $\lambda x \in V_a(S)$. □

(b) $X \subseteq \mathbb{A}^{n+1}$ cone $\Rightarrow I(X) \trianglelefteq K[x_0,\dots,x_n]$ homogeneous.

Proof $f = \sum_{d\in\mathbb{N}} f_d \in I(X) \xrightarrow{[\text{Lem } 6.10(a)]}$ suffices to show: $f_d \in I(X)\ \forall d$

Let $x \in X$, $\lambda \in K \rightsquigarrow \lambda x \in X$ (X cone)

$$\Rightarrow \underset{f\in I(X)}{0 =} f(\lambda x) = \underbrace{\sum_{d\in\mathbb{N}} \lambda^d \cdot f_d(x)}_{\substack{\text{see as polynomial in } K[\lambda] \\ \text{vanishes at all } \lambda\in K \xrightarrow[\text{closed}]{K \text{ alg.}} f_d(x)=0\ \forall d}} \quad \forall \lambda \in K.$$

$\square$

Lem (Cones $\leftrightarrow$ projective varieties)
There is a bijection

$$\{\text{cones in } \mathbb{A}^{n+1}\} \xleftrightarrow{1:1} \{\text{Projective varieties in } \mathbb{P}^n\}$$
$$X \longmapsto \mathbb{P}(X)$$
$$C(Y) \longmapsfrom Y$$

Proof $\boxed{\text{Well-definedness}}$

$S = \{f \in J : J \text{ homogen.}\}$
$\rightsquigarrow J = \langle S \rangle$

• $X \subseteq \mathbb{A}^{n+1}$ cone $\overset{\text{Rmk}}{\Longrightarrow} J = I_a(X)$ homog. ideal and $X = V_a(J)$
$= V_a(S)$

$\Rightarrow \mathbb{P}(X) = \{(x_0:\dots:x_n) \in \mathbb{P}^n : \underbrace{(x_0,\dots,x_n) \in X}_{\Leftrightarrow f(x)=0\ \forall f\in S}\} = V_p(J)$ projective variety

• $Y \subseteq \mathbb{P}^n$ proj. variety, $Y = V_p(J)$, $J \leq K[x_0,\dots,x_n]$ homogen. ideal

$$\Rightarrow C(Y) = \{0\} \cup \{(x_0,\dots,x_n) \in \mathbb{A}^{n+1}\setminus\{0\} : \underbrace{(x_0:\dots:x_n) \in Y}_{\Leftrightarrow f(x)=0\ \forall f\in J \text{ homogen.}}\}$$

$$= \begin{cases} \{0\} & \text{if } J \text{ contains non-zero const. polynom.} \\ \underset{\text{cone by Rmk above}}{V_a(J)} & \text{otherwise} \end{cases}$$

$\boxed{\text{Bijection}}$ Notation as above

• $C(\mathbb{P}(X)) = C(V_p(\underbrace{I_a(X)}_{\substack{X\neq\emptyset \rightsquigarrow I_a(X) \text{ contains} \\ \text{no non-zero const. poly}}})) = V_a(I_a(X)) = X$

• $\mathbb{P}(C(Y)) = \pi(\pi^{-1}(Y)) = Y$ $\quad \pi : \mathbb{A}^{n+1}\setminus\{0\} \twoheadrightarrow \mathbb{P}^n$ $\square$

The projective Nullstellensatz

<u>Would like</u>: $V_p(I_p(X)) = X$, $I_p(V_p(J)) = \sqrt{J}$ for X proj. variety, J homog. ideal

<u>Idea</u> Prove by correspondence to cones & usual Nullstellensatz

<u>Problem</u> Special cases in previous proofs when $J = K[x_0, \dots, x_n]$

$\rightsquigarrow X = V_p(J) = \emptyset \rightsquigarrow C(\emptyset) = \{0\} \rightsquigarrow I_a(\{0\}) = \langle x_0, \dots, x_n \rangle$

this ideal needs special treatment!

<u>Indeed</u>: $I_p(\underbrace{V_p(\langle x_0, \dots, x_n \rangle)}_{= \emptyset \text{ since } (0:0:\dots:0) \text{ not allowed in } \mathbb{P}^n}) = I_p(\emptyset) = K[x_0, \dots, x_n] \neq \sqrt{\langle x_0, \dots, x_n \rangle}$!

Def (Irrelevant ideal)

The (radical homogeneous) ideal $I_0 = \langle x_0, \dots, x_n \rangle \trianglelefteq K[x_0, \dots, x_n]$ is called the <u>irrelevant ideal</u>.

Pro (Projective Nullstellensatz)

(a) X projective variety $\Rightarrow V_p(I_p(X)) = X$.

(b) $J \trianglelefteq K[x_0, \dots, x_n]$ homogeneous w/ $\sqrt{J} \neq I_0 \Rightarrow I_p(V_p(J)) = \sqrt{J}$

In particular, there is an inclusion-reversing bijection

$$\left\{ \begin{array}{c} \text{projective} \\ \text{varieties in } \mathbb{P}^n \end{array} \right\} \underset{V_p(J) \leftarrow\!\shortmid J}{\overset{X \mapsto I_p(X)}{\rightleftarrows}} \left\{ \begin{array}{c} \text{homogeneous radical ideals} \\ \text{in } K[x_0, \dots, x_n] \text{ not equal to } I_0 \end{array} \right\}$$

Proof (a) and (b) "$\supseteq$" and inclusion-reversing property of $V_p(-), I_p(-)$

$\rightsquigarrow$ follow exactly like for affine varieties.

$\boxed{\text{(b) "}\subseteq\text{"}}$ J homogeneous & $\sqrt{J} \neq I_0$.

If J contains non-zero const. poly $\Rightarrow J = K[x_0, \dots, x_n] \Rightarrow I_p(V_p(J)) \subseteq \sqrt{J}$ ✓

Assume otherwise and let $X = V_a(J) \subseteq \mathbb{A}^{n+1}$, $Y = V_p(J) \subseteq \mathbb{P}^n$

- Proof of Lem (Cones $\leftrightarrow$ Proj. var.): $X = C(Y) \Rightarrow Y = \mathbb{P}(X)$
- Nullstellensatz: $I_a(X) = \sqrt{J} \neq I_0 = I_a(\{0\})$

$\xrightarrow[\text{aff. varieties} \leftrightarrow \text{rad. ideals}]{\text{bijection}} X \neq \{0\} \xrightarrow[\text{cones} \leftrightarrow \text{proj. var.}]{\text{biject.}} Y \neq \emptyset$

$$I_p(Y) = \langle f \in K[x_0,\dots,x_n] \text{ homog.} : f(x)=0 \ \forall x \in Y \rangle$$
$$= \langle f \in K[x_0,\dots,x_n] \text{ homog.} : \underbrace{f(x)=0 \ \forall x \in X\setminus\{0\}} \rangle$$

Claim then also $f(0)=0$
↳ $\deg(f)>0 \rightsquigarrow$ automatic
↳ $\deg(f)=0 \rightsquigarrow f=0$ since $X\setminus\{0\} \neq \emptyset$

$$\Rightarrow I_p(Y) = \underbrace{I_a(X)}_{\substack{\text{generated by its}\\ \text{homogeneous elements}}} \overset{\text{above}}{=} \sqrt{J}$$

$\boxed{\text{Bijection}}$

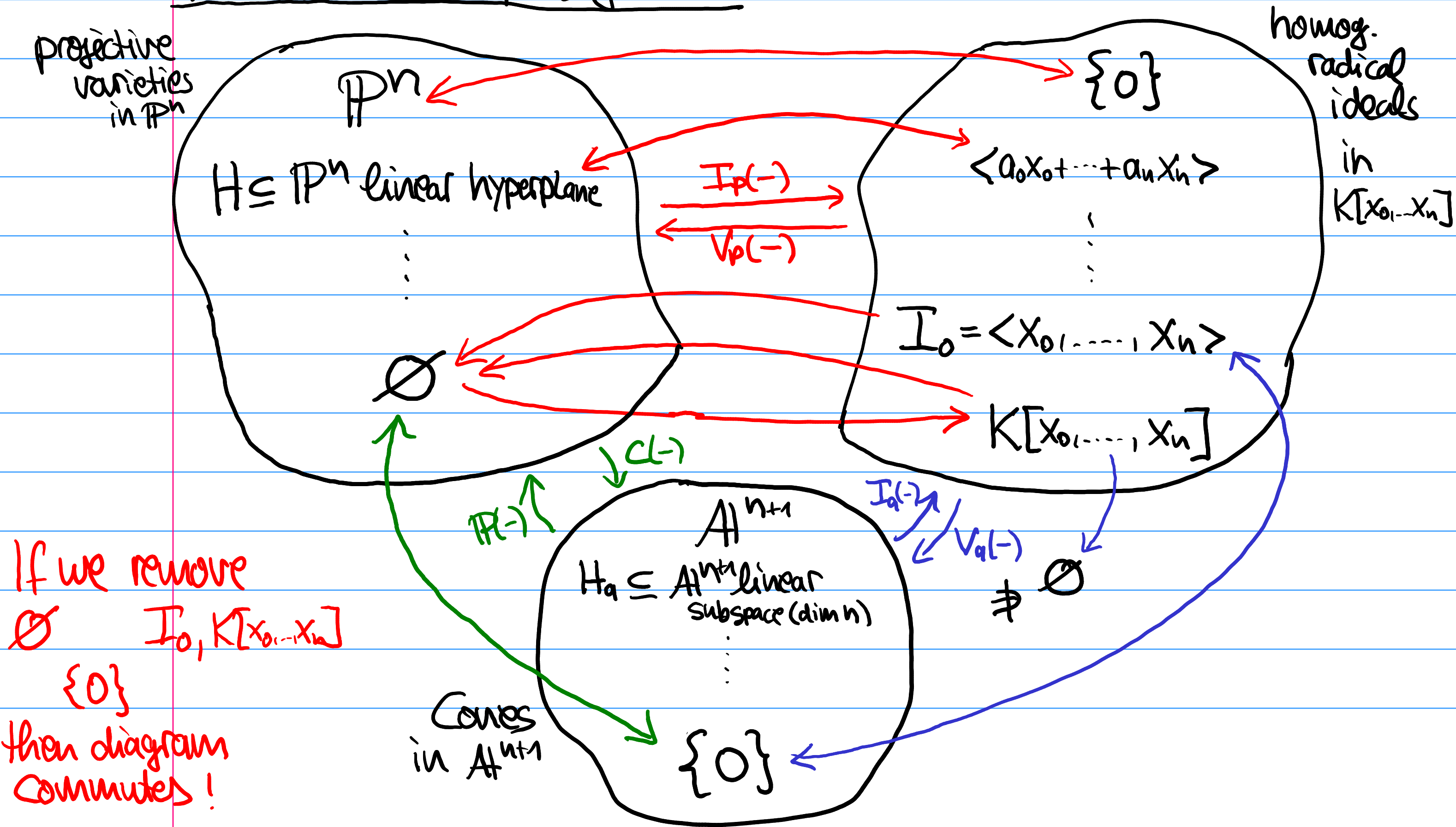
$$\left\{ \begin{matrix} \text{projective} \\ \text{varieties in } \mathbb{P}^n \end{matrix} \right\} \underset{V_p(J) \leftarrow\!\shortmid J}{\overset{X \mapsto I_p(X)}{\rightleftarrows}} \left\{ \begin{matrix} \text{homogeneous radical ideals} \\ \text{in } K[x_0,\dots,x_n] \text{ not equal to } I_0 \end{matrix} \right\}$$

$$X = V_p(J) \subseteq \mathbb{P}^n \text{ projective} \Rightarrow I_p(X) = \sqrt{J} \text{ radical and } \underbrace{I_p(X) \neq I_0}_{\text{otherwise}: X = V_p(I_p(X))}$$

$= V_p(\langle x_0,\dots,x_n\rangle) = \emptyset$ but $I_p(\emptyset) = K[x_0,\dots,x_n]$ ↯

$\rightsquigarrow$ (a) & (b) $\Rightarrow$ $V_p(-)$ and $I_p(-)$ inverse maps □

Illustration of bijection



Rmk (Properties of V_P and I_P) $\rightsquigarrow$ same proofs as affine case.

(a) $S_1, S_2, S_i \subseteq K[x_0, \ldots, x_n]$ sets of hom. polynomials ($i \in I$)

$$\Rightarrow V_P(S_1) \cup V_P(S_2) = V_P(S_1 \cdot S_2)$$

$$\bigcap_{i \in I} V_P(S_i) = V_P\Big(\bigcup_{i \in I} S_i\Big)$$

(b) $J_1, J_2 \trianglelefteq K[x_0, \ldots, x_n]$ hom. ideals

$$\Rightarrow V_P(J_1) \cup V_P(J_2) = V_P(J_1 \cdot J_2) = V_P(J_1 \cap J_2)$$

$$V_P(J_1) \cap V_P(J_2) = V_P(J_1 + J_2)$$

(c) $X_1, X_2 \subseteq \mathbb{P}^n$ projective varieties

$$\Rightarrow I_P(X_1 \cup X_2) = I_P(X_1) \cap I_P(X_2)$$

$\sqrt{I_P(X_1) + I_P(X_2)} = I_0$

$$I_P(X_1 \cap X_2) = \begin{cases} R[x_0, \ldots, x_n], & \text{if } X_1, X_2 \text{ disjoint} \\ \sqrt{I_P(X_1) + I_P(X_2)}, & \text{otherwise} \end{cases}$$

Homogeneous coordinate rings, projective subvarieties and the Zariski topology

Def (Homog. coord. rings)
$Y \subseteq \mathbb{P}^n$ projective variety

$$\Rightarrow \quad S(Y) := K[x_0,\dots,x_n] / \underbrace{I(Y)}_{\text{homogeneous}} \quad \text{homog. coord. ring}$$

$\overset{\text{Lem}}{\Longrightarrow}$ $S(Y)$ graded ring

Exa • $S(\mathbb{P}^1) = K[x_0, x_1]$
• $Y = V(x_0+x_1+x_2) \subseteq \mathbb{P}^2$ linear subspace
$\Rightarrow S(Y) = K[x_0,x_1,x_2]/\langle x_0+x_1+x_2\rangle \cong K[x_0,x_1]$
later: Y isom. to $\mathbb{P}^1$!

Note • Elements of $S(Y)$ still don't make sense as functions $Y \to K$
(e.g. $f = x_0 \in S(\mathbb{P}^1) \rightsquigarrow 1 = f(1:1) \neq f(2:2) = 2$)

• $f \in S(Y)$ homogeneous $\rightsquigarrow V_Y(f) = \{x \in Y : f(x) = 0\}$ well-def.!
has homogeneous representative in $K[x_0,\dots,x_n]$

Construction (Relative versions of $V_p(-)$ and $I_p(-)$)
$Y \subseteq \mathbb{P}^n$ projective variety
• $J \trianglelefteq S(Y)$ homogeneous ideal

$$\Rightarrow \quad V_Y(J) := V_{Y,p}(J) := \{x \in Y : f(x) = 0 \ \forall \text{homog. } f \in J\}$$

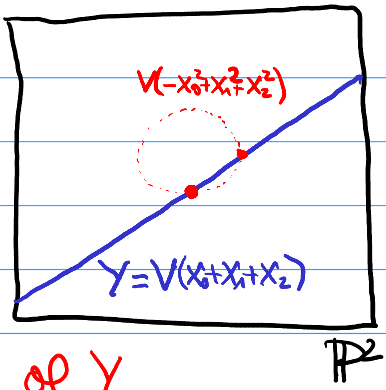
similar: $V_Y(S)$

• $X \subseteq Y$ subset

$$\Rightarrow \quad I_Y(X) := I_{Y,p}(X) := \langle f \in S(Y) \text{ homog.} : f(x) = 0 \ \forall x \in X\rangle \subseteq S(Y)$$

Subsets $X = V_Y(J) \subseteq Y$: projective subvarieties of Y
$\leftrightarrow$ project. varieties of $\mathbb{P}^n$ cont. in Y.

Exa $Y = V(X_0+X_1+X_2) \subseteq \mathbb{P}^2_{\mathbb{C}}$



$\rightsquigarrow V_Y(-X_0^2+X_1^2+X_2^2)$

$= V(X_0+X_1+X_2,\ \underbrace{-(-X_1-X_2)^2+X_1^2+X_2^2}_{= \underbrace{-2X_1X_2}_{=0 \Leftrightarrow X_1=0 \text{ or } X_2=0}})$

$= \{(1:0:-1),\ (1:-1:0)\} \rightsquigarrow$ subvariety of Y

Rmk

- $\exists$ relative version of Nullstellensatz for $V_Y(-)$ and $I_Y(-)$.
- sometimes useful:
  Every proj. subvariety $X \subseteq Y \subseteq \mathbb{P}^n$ can be written as zero locus of fin. many homogeneous poly's in $S(Y)$ of the same degree.

  Reason: $V_P(f) = \underbrace{V_P(X_0^d f, \ldots, X_n^d f)}_{= \underbrace{V_P(\langle X_0^d, \ldots, X_n^d\rangle)}_{=\emptyset} \cup V_P(f)}$ f homog., $d \in \mathbb{N}$
- Properties of $V_Y(-)$ $\Rightarrow$ finite unions & arbitr. intersections of subvar. of Y are subvarieties.

Def (Zariski topology)
Zariski topology on projective variety Y:

$X \subseteq Y$ closed $\Leftrightarrow$ X subvariety of Y.

Note Zar. top. on $Y \subseteq \mathbb{P}^n$ = relative top. of Zar. top on $\mathbb{P}^n$

Homogenization and dehomogenization

Recall

→ Zariski topology on $\mathbb{A}^n$ and $\mathbb{P}^n$ (closed sets = affine / project. subvar.)
& have: $\mathbb{A}^n \subseteq \mathbb{P}^n$ as set
⇒ **Q** Is Zar. top. on $\mathbb{A}^n$ induced from $\mathbb{P}^n$?

→ on algebraic side: want to relate

polynomials in $x_1, \dots, x_n$ $\longleftrightarrow$ homogeneous polynomials in $x_0, \dots, x_n$

Construction (Homogenization and dehomogenization)

(a) $f \in K[x_0, \dots, x_n]$ homogeneous

$\rightsquigarrow f^i := \underbrace{f(x_0 = 1)}_{\text{works also for } f \text{ not homogeneous!}} \in K[x_1, \dots, x_n]$ <u>dehomogenization of f</u>

Note

- setting $x_0 = 1$ is (surjective) ring homomorphism:
  $(f+g)^i = f^i + g^i$ and $(f \cdot g)^i = f^i \cdot g^i$
- this implies that we can do the same for ideals:
  $J \trianglelefteq K[x_0, \dots, x_n]$ homogeneous

$\rightsquigarrow J^i := \{f^i : f \in J\} \subseteq K[x_1, \dots, x_n]$ is ideal

(b) Let $f = \sum_{i_1, \dots, i_n \in \mathbb{N}} a_{i_1, \dots, i_n} \cdot x_1^{i_1} \cdots x_n^{i_n} \in K[x_1, \dots, x_n]$
↘ $f \neq 0$ of degree d

$\rightsquigarrow f^h := x_0^d \cdot f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$ <u>homogenization of f</u>

$= \sum_{i_1, \dots, i_n \in \mathbb{N}} a_{i_1, \dots, i_n} \cdot x_0^{d - i_1 - \dots - i_n} \cdot x_1^{i_1} \cdots x_n^{i_n}$ → homogeneous of degree d

Note

- $f, g \in K[x_1, \dots, x_n]$ degrees d, e ⇒ $f \cdot g$ degree $d+e$ and
  $(f \cdot g)^h = x_0^{d+e} \cdot f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \cdot g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = f^h \cdot g^h$
- However: $(f+g)^h \neq f^h + g^h$ (see below)

$\rightsquigarrow J \trianglelefteq K[x_1, \dots, x_n]$ ideal $\rightsquigarrow J^h = \langle f^h : 0 \neq f \in J \rangle$ ← need to take ideal
↖ homogeneous ideal

Exa $\cdot\ f = x_1^2 - x_2^2 - 1 \rightsquigarrow f^h = x_0^2 \cdot \left(\left(\frac{x_1}{x_0}\right)^2 - \left(\frac{x_2}{x_0}\right)^2 - 1 \right)$
$= x_1^2 - x_2^2 - x_0^2$

$\rightsquigarrow (f^h)^i = x_1^2 - x_2^2 - x_0^2 \big|_{x_0=1} = f.$

$\cdot$ Note: $f = \underbrace{x_1^2 - x_2^2}_{=: g} \underbrace{-1}_{=: \tilde{g}} \rightsquigarrow g^h = g,\ \tilde{g}^h = \tilde{g}$ but $f^h = (g+\tilde{g})^h$
$\neq g + \tilde{g} = f.$

Exercise Prove or provide a counterexample

(a) $S \subseteq K[x_0, \dots, x_n]$ set of homogeneous polynomials
$\Rightarrow \langle S \rangle^i = \langle S^i \rangle$ with $S^i = \{ f^i : f \in S \}$

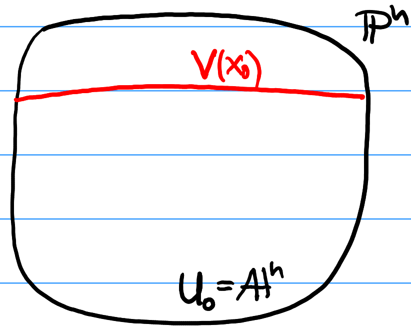
(b) $S \subseteq K[x_1, \dots, x_n]$ set of nonzero polynomials
$\Rightarrow \langle S \rangle^h = \langle S^h \rangle$ with $S^h = \{ f^h : f \in S \}.$

Rmk ($\mathbb{A}^n$ as an open subset of $\mathbb{P}^n$)

Have subset

$$\mathbb{A}^n \xrightarrow[\sim]{F} U_0 = \{ (x_0 : \dots : x_n) \in \mathbb{P}^n : x_0 \neq 0 \} \subseteq \mathbb{P}^n$$
bijection
$$(x_1, \dots, x_n) \longmapsto (1 : x_1 : \dots : x_n)$$

Note $U_0 \subseteq \mathbb{P}^n$ open: $\mathbb{P}^n \setminus U_0 = V(x_0)$ linear hyperplane "at infinity"



Claim Subspace topology on $\mathbb{A}^n \subseteq \mathbb{P}^n$
$=$ Zariski top. on $\mathbb{A}^n$

Pf

(a) $X = V_p(J) \cap \mathbb{A}^n$ closed in subspace top. $J \trianglelefteq K[x_0, \dots, x_n]$ homogeneous

$\Rightarrow X = V_a(J^i)$ Zariski closed

$f \in J$ homogen. $\Rightarrow f(F(x_1, \dots, x_n)) = f^i(x_1, \dots, x_n)$

(b) $X = V_a(J) \subseteq \mathbb{A}^n$ Zariski closed $J \trianglelefteq K[x_1, \dots, x_n]$

$\Rightarrow X = V_p(J^h) \cap \mathbb{A}^n$ closed in subspace topology

$g \in J \Rightarrow g^h(F(x_1, \dots, x_n)) = g(x_1, \dots, x_n)$

$\Rightarrow F : \mathbb{A}^n \to U_0$ homeomorphism

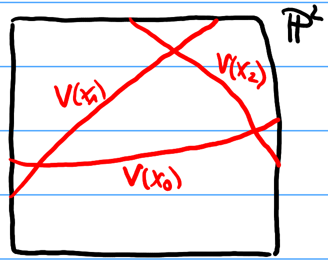
later: isomorphism of varieties.

Topological properties of projective space

Know $\mathbb{A}^n$ is irreduc. of dimension n $\rightsquigarrow$ properties of $K[x_1,\dots,x_n]$

Pro (Irreducibility and dimension of $\mathbb{P}^n$)
Projective space $\mathbb{P}^n$ is irreducible of dimension n.

Pf Have seen: $\mathbb{A}^n \cong U_0 = \{(x_0:\dots:x_n)\in\mathbb{P}^n : x_0 \neq 0\} \subseteq \mathbb{P}^n$ open subset
$\uparrow$ irreducible



Similar: $\mathbb{A}^n \cong U_i = \{(x_0:\dots:x_n)\in\mathbb{P}^n : x_i \neq 0\} \subseteq \mathbb{P}^n$ $\forall i$

We have: $\mathbb{P}^n = \underbrace{U_0 \cup \dots \cup U_n}_{\text{cover by open, irreducible sets, dim } n} \rightarrow U_i \cap U_j \neq \emptyset \ \forall i,j$

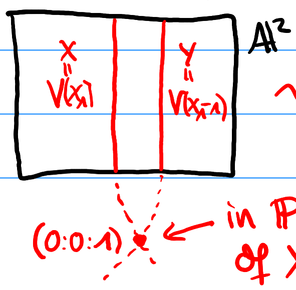
$\Rightarrow \mathbb{P}^n$ is irreducible [Exercise 2.21(b)] of dimension n [Exercise 2.34(a)]. $\square$

Know X affine variety.
X is irreducible $\Leftrightarrow A(X)$ is an integral domain

Exercise Show: X projective variety.
X is irreducible $\Leftrightarrow S(X)$ is an integral domain
Hint: Prove that for R graded ring we have:
R is an integral domain $\Leftrightarrow \forall f,g \in R$ homogeneous w/ $f\cdot g = 0$
we have $f=0$ or $g=0$.

Exercise [see Exerc. 6.31 for hints]
$X, Y \subseteq \mathbb{P}^n$ non-empty projective varieties.
If $\underbrace{\dim X + \dim Y \geq n}_{}$, then $X \cap Y \neq \emptyset$.
for X, Y equidimensional
$\Leftrightarrow \operatorname{codim}_{\mathbb{P}^n} X + \operatorname{codim}_{\mathbb{P}^n} Y \leq n$

Note: Corresponding statement is false for affine space:



$\rightsquigarrow \dim X + \dim Y = 1 + 1 \geq 2$ but $X \cap Y = \emptyset$.

$\leftarrow$ in $\mathbb{P}^2$ the closures of X, Y meet on line at ∞.

Projective closure

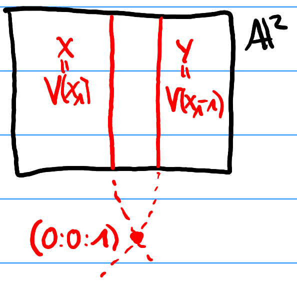
Q Given an affine variety $X \subseteq \mathbb{A}^n$, what is the Zariski closure

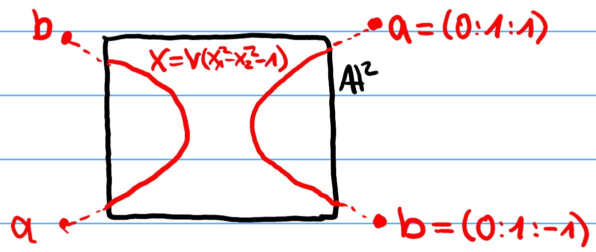
$$\overline{X} \subseteq \mathbb{P}^n \quad ?$$

$\mathbb{A}^n \subseteq \mathbb{P}^n$ as U_0

$\rightsquigarrow$ projective closure of X

Exa





Candidate $X = V_a(J) \subseteq \mathbb{A}^n$, $J \trianglelefteq K[x_1, \dots, x_n]$ ideal

$$\Rightarrow \quad X = \underbrace{V_p(J^h)}_{\text{Zar. closed set containing } X} \cap \mathbb{A}^n$$

Pro (Computation of projective closure)
$X = V_a(J) \subseteq \mathbb{A}^n$, $J \trianglelefteq K[x_1, \dots, x_n]$ ideal.

(a) We have:
$$\overline{X} = V_p(J^h) \subseteq \mathbb{P}^n.$$

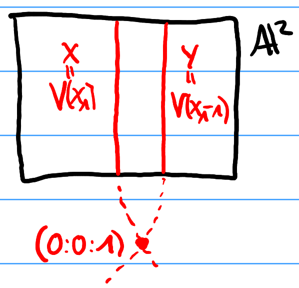
(b) If $J = \langle f \rangle$ is a non-zero principal ideal, then
$$\overline{X} = V_p(f^h) \subseteq \mathbb{P}^n.$$

Exa • $Y = V_a(x_1 - 1) \subseteq \mathbb{A}^2$

$\Rightarrow \ \overline{Y} = V_p(x_1 - x_0) \subseteq \mathbb{P}^2$

Added points on line $L = V_p(x_0)$ at ∞:

$\overline{Y} \cap L = V_p(x_1 - x_0, x_0) = \{(0:0:1)\}$

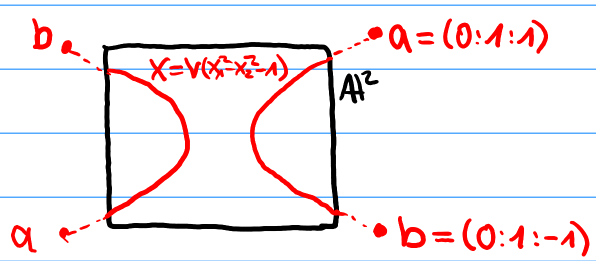


• $X = V_a(x_1^2 - x_2^2 - 1) \subseteq \mathbb{A}^2$

$\Rightarrow \overline{X} = V_p(x_1^2 - x_2^2 - x_0^2) \subseteq \mathbb{P}^2$

$\overline{X} \cap L = V_p(x_1^2 - x_2^2 - x_0^2, x_0)$

$= V_p(\underbrace{x_1^2 - x_2^2}_{(x_1-x_2)(x_1+x_2)}, x_0) = \{\overset{x_1=x_2}{(0:1:1)}, \underset{x_1=-x_2}{(0:1:-1)}\}$



Proof of Prop.

(a) Want to show: $X = V_a(J) \Rightarrow \overline{X} = V_P(J^h)$.

By def: $\overline{X}$ = smallest closed subset containing X

Know: $V_P(J^h)$ is a closed subset containing $X \Rightarrow \overline{X} \subseteq V_P(J^h)$.

Conversely, let $Y = V_P(J') \supseteq X$ be closed in $\mathbb{P}^n$.

$[J' \trianglelefteq K[x_0, \ldots, x_n]$ homogeneous

Want to show: $V_P(J^h) \subseteq Y \Leftrightarrow J' \subseteq \sqrt{J^h}$

Nullstellensatz (apply $I_P(-)$ on both sides)

J' generated by its homogeneous elements $g \in J'$

Claim $g = x_0^d \cdot f^h$ for some $d \in \mathbb{N}$, $f \in K[x_1, \ldots, x_n]$

[Exa $g(x_0, x_1, x_2) = x_0^2 x_1^3 + 2 x_0^4 x_2 \Rightarrow g = x_0^2 \cdot (x_1^3 + 2x_2)^h$]

Want to show: $g \in \sqrt{J^h}$ (wlog: $g \neq 0$)

Indeed:

$g = x_0^d f^h$ is zero on $X \subseteq \mathbb{P}^n$	$(X \subseteq Y)$
$\Rightarrow f = (f^h)^i$ is zero on $X \subseteq \mathbb{A}^n$	$(x_0 \neq 0$ on $X \subseteq \mathbb{A}^n)$
$\Rightarrow f \in I_a(X) = I_a(V_a(J)) = \sqrt{J}$	(Nullstellensatz)
$\Rightarrow f^m \in J$ for some $m \in \mathbb{N}$	(Def. of radical ideal)
$\Rightarrow (f^h)^m = (f^m)^h \in J^h$ for some $m \in \mathbb{N}$	(Construct. of J^h)
$\Rightarrow f^h \in \sqrt{J^h}$	
$\Rightarrow x_0^d \cdot f^h = g \in \sqrt{J^h}$	##

(b) $J = \langle f \rangle = \{ f \cdot g : g \in K[x_1, \ldots, x_n] \}$

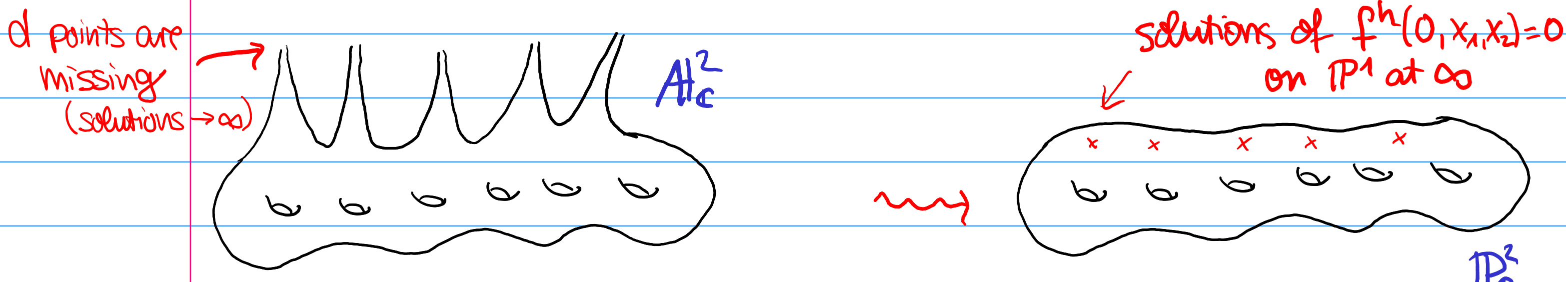
$\Rightarrow J^h \overset{\text{def}}{=} \langle \underbrace{(f \cdot g)^h}_{= f^h \cdot g^h} : g \in K[x_1, \ldots, x_n] \rangle = \langle f^h \rangle$

"$\subseteq$": take $g = 1$

"$\supseteq$" $f^h \cdot g^h \in \langle f^h \rangle$.

$\square$

Note Projective closure is behind some of the pictures I drew earlier:



$V_a(f)$, $f \in \mathbb{C}[x_1,x_2]_d$ $\qquad\qquad$ $V_p(f^h)$

Projective hypersurfaces

Def A projective variety $X \subseteq \mathbb{P}^n$ is a <u>hypersurface</u> if X is of pure dimension $n-1$.

For affine hypersurf. $Y \subseteq \mathbb{A}^n$ we have seen:
$I_a(Y) = \langle f \rangle$ principal $\rightsquigarrow$ define $\deg(Y) = \deg(f)$.

Pro/Def $X \subseteq \mathbb{P}^n$ hypersurface
$\Rightarrow I_p(X) = \langle f \rangle$ principal $\rightsquigarrow$ define $\deg(X) = \deg(f)$.

Proof First prove case: X does not contain $V_p(x_0)$ (as irred. component)
$\Rightarrow Y := X \cap \mathbb{A}^n$ affine hypersurface with $\overline{Y} = X$
["$\subseteq$": $Y \subseteq X \Rightarrow \overline{Y} \subseteq X$
"$\supseteq$": $X = X_1 \cup \dots \cup X_r$ irred. decomposition, then $\forall i : X_i \nsubseteq V_p(x_0)$ ↑ otherwise: $X_i = V_p(x_0)$ for dim. reasons
$\Rightarrow \emptyset \neq X_i \cap \mathbb{A}^n \subseteq X_i$ open
$\overset{X_i \text{ irred.}}{\underset{[\text{Rmk } 2.12]}{\Longrightarrow}} X_i \cap \mathbb{A}^n$ dense in X_i $\Rightarrow Y = \bigcup_{i=1}^{r} X_i \cap \mathbb{A}^n$ dense in X.]

We have: $Y = V_a(f) \overset{\text{Pro}}{\Longrightarrow} X = V_p(f^h)$. cut out by single eqn.

General case: $X = \underbrace{X_1 \cup \dots \cup X_{r-1}}_{=:X'} \cup V_p(x_0)$ irred. decomp.

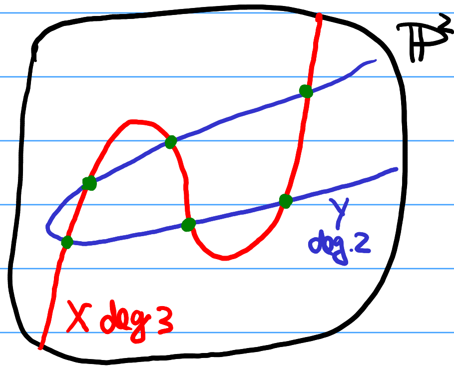
$\overset{\text{Case above}}{\Longrightarrow} X' = V_p(\tilde{f}) \Rightarrow X = V_p(x_0 \cdot \tilde{f})$.

In general: Can assume $X = V(f)$, $f \in K[x_0, \dots, x_n]$ homog.
For
$f = f_1^{e_1} \cdot f_2^{e_2} \cdot \ldots \cdot f_r^{e_r}$ irred. decompos. in $K[x_0, \dots, x_n]$
Exercise: f_i are homogeneous and $I_p(X) = \sqrt{\langle f \rangle} = \langle f_1 \cdot \ldots \cdot f_r \rangle$
↑ principal. □

Def $X \subseteq \mathbb{P}^n$ hypersurface is (linear) hyperplane, quadric, cubic if it is of degree 1, 2, 3, respectively.

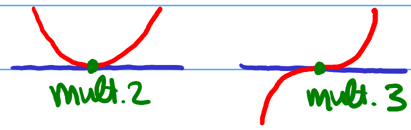
Cool result (we don't have time to discuss):

Thm (Bézout)
$X, Y \subseteq \mathbb{P}^2$ hypersurfaces of degrees d, e, not sharing any irred. components.



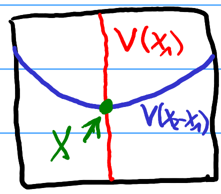
$\Rightarrow$ X and Y intersect in $d \cdot e$ points counted with multiplicity.

e.g.



Warning Have seen: $X = V_a(\langle f \rangle)$ hypersurf. $\Rightarrow \overline{X} = V_p(\langle f^h \rangle)$
In general: homogeniz. of ideal not generated by homogenization of generators !

Exa $J = \langle x_1, x_2 - x_1^2 \rangle \trianglelefteq K[x_1, x_2]$



$\rightsquigarrow X = V_a(J) = \{(0,0)\} \subseteq \mathbb{A}^2$
$\rightsquigarrow$ X also closed in $\mathbb{P}^2$:
$\overline{X} = \{(1:0:0)\}$

Homogenize generators:
$$V_p\left((x_1)^h, (x_2 - x_1^2)^h\right)$$
$$= V_p(x_1, x_0 x_2 - x_1^2)$$
$$= \{(1:0:0), (0:0:1)\}.$$

$\rightsquigarrow$ can be fixed by computing a Gröbner basis

$$\underbrace{J = \langle f_1, \dots, f_r \rangle}_{f_i \text{ must satisfy extra condit. to be a Gröbner basis}} \rightsquigarrow J^h = \langle f_1^h, \dots, f_r^h \rangle$$

in principle: $\exists$ computer algorithm to calculate J^h.